2.1 Matrix Operations

Key idea: We can extend much of the algebra of the real numbers to matrices (much like we did for vectors) including multiplication: we will see this does not fully extend as commutativity fails and only some matrices have multiplicative inverses (so some idea of division but not a full generalization).

"Matrices have already shown their utility. In chapter 2 we investigate these objects in detail for their own interest. We will see that this study is fruitful for our previous and future topics: to begin, we simply investigate the algebraic operations on matrices."

Def: An $m \times n$ matrix is a rectangular array of numbers (or functions in some settings) in m rows and n columns. The (i,j)-entry is the i^{th} entry of j^{th} column where $1 \le i \le m$ and $1 \le j \le n$.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & & a_{2j} & & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix} = \left[a_{ij} \right]_{\substack{1 \le i \le m \\ 1 \le j \le n}}$$

Mnemonic: a_{ij}
j is on right
columns left to right.

- The j^{th}-column of A is the vector $\vec{a}_j = [a_{ij}]_{1 \le i \le m} = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix}$

- The diagonal entries of A are $a_{11}, a_{22}, a_{33}, a_{44}, \dots$ If these are the only non-zero entries of a <u>square</u> matrix A then A is a diagonal matrix.

- A is a zero matrix if all $a_{ij} = 0$.

- The transpose of A is the matrix A^T whose j^{th} row is the j^{th} column of A.

$$\text{So if } A = [a_{ij}], \text{ then } A^T = [a_{ji}].$$

(We go ahead and define basic algebraic operations on matrices, these ideas are probably familiar but perhaps not in this notation.)

- Two $m \times n$ matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are equal, written $A = B$ if all $a_{ij} = b_{ij}$.

- The sum of two $m \times n$ matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ is the matrix:

$$A + B = [a_{ij} + b_{ij}].$$

- The scalar multiple of $A = [a_{ij}]$ by the scalar c is the matrix:

$$cA = c[a_{ij}] = [ca_{ij}].$$

<u>Ex</u> Let
$$A = \begin{bmatrix} 1 & 6 \\ 1 & 2 \\ 3 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 3 \\ 0 & 1 \\ 1 & 2 \end{bmatrix}, \quad C = \begin{bmatrix} 3 & 1 & 4 \\ 3 & 3 & 5 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

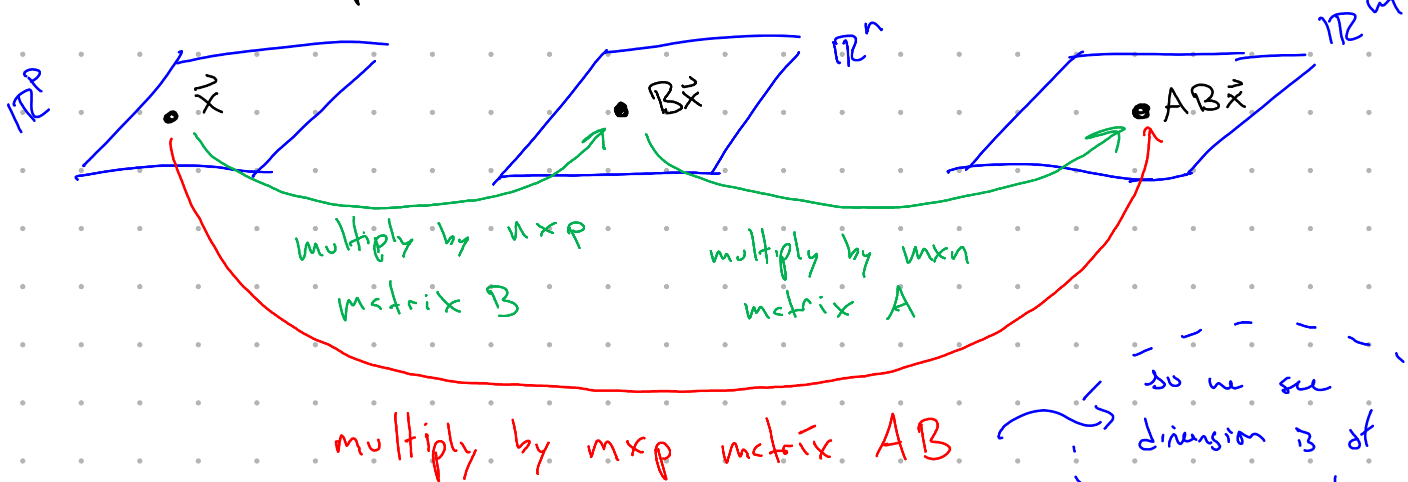
Notice the dimensions / columns. Compute a_{32} say. D is diagonal.

$A + B$, $B + C$ not defined, $A + B = C^T$, $D^T = D$, $3A$, $-\frac{1}{2}B$, ... etc.

We've seen little that's new, but now we consider the most important operation for our interests in this course.

Matrix Multiplication:

We've previously seen that matrices transforms vectors by way of a product: A $m\times n$, $\vec{x}$ in $\mathbb{R}^n$ $\longmapsto A\vec{x}$ in $\mathbb{R}^m$. This motivates our definition of the matrix product AB.



so we see dimension is of concern in matrix mult.!

This picture defines the matrix product:

Def: If A is an $m\times n$ matrix, B is an $n\times p$ matrix then the product of A and B is the $m\times p$ matrix:

$$AB = \left[A\vec{b}_1 \;\; A\vec{b}_2 \;\; \cdots \;\; A\vec{b}_p\right]$$

Note: each $A\vec{b}_i$ is a vector (column) in $\mathbb{R}^m$

Why this definition? If $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_p \end{bmatrix}$ is in $\mathbb{R}^p$, then

$$B\vec{x} = [\vec{b}_1 \;\; \vec{b}_2 \;\cdots\; \vec{b}_p]\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{bmatrix} = x_1\vec{b}_1 + x_2\vec{b}_2 + \cdots + x_p\vec{b}_p$$ ⟵ a vector in $\mathbb{R}^n$

$$\Rightarrow AB\vec{x} = A(x_1\vec{b}_1 + \cdots + x_p\vec{b}_p) = A(x_1\vec{b}_1) + A(x_2\vec{b}_2) + \cdots + Ax_p\vec{b}_p$$

$$= x_1A\vec{b}_1 + x_2A\vec{b}_2 + \cdots + x_pA\vec{b}_p$$

⟶ a vector in $\mathbb{R}^m$

$$(AB)\vec{x} = \left[A\vec{b}_1 \;\; A\vec{b}_2 \;\cdots\; A\vec{b}_p\right]\begin{bmatrix} x_1 \\ \vdots \\ x_p \end{bmatrix}.$$

The key idea here is matrix multiplication is just the composition of linear transformations (see video on webpage).

As with any arithmetic idea, this requires practice.

Ex Let $A = \underbrace{\begin{bmatrix} 1 & 0 \\ 2 & 2 \\ 3 & -1 \end{bmatrix}}_{3 \times 2}$ and $B = \underbrace{\begin{bmatrix} 1 & 2 & 1 & 0 \\ 1 & 2 & -1 & 1 \end{bmatrix}}_{2 \times 4}$. Compute AB.

3×2 ⟵ match! ⟶ 2×4

$$A\vec{b}_1 = \begin{bmatrix} 1 & 0 \\ 2 & 2 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix}, \quad A\vec{b}_2 = \begin{bmatrix} 1 & 0 \\ 2 & 2 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 8 \\ 4 \end{bmatrix},$$

$$A\vec{b}_3 = \begin{bmatrix} 1 & 0 \\ 2 & 2 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}, \quad A\vec{b}_4 = \begin{bmatrix} 1 & 0 \\ 2 & 2 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}$$

So,
$$AB = \begin{bmatrix} A\vec{b}_1 & A\vec{b}_2 & A\vec{b}_3 & A\vec{b}_4 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 2 & 1 & 0 \\ 4 & 8 & 0 & 2 \\ 2 & 4 & 4 & -1 \end{bmatrix}}_{3 \times 4}$$

You may have encountered an equivalent definition of AB:

"Def:" the (i,j)-entry of AB is $a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$.

we use entries of the i^{th} row of A
and entries of the j^{th} column of B

Ex Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}$. Compute AB and BA.

$$AB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 5 \\ -4 & 11 \end{bmatrix}$$

$$BA = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}$$

__Notice!__: $AB \neq BA$.

This and other facts are posted on the webpage

Basic algebra properties

Let A, B, and C be matrices of the same size, and let r and s be scalars.

a. $A + B = B + A$
b. $(A + B) + C = A + (B + C)$
c. $A + 0 = A$
d. $r(A + B) = rA + rB$
e. $(r + s)A = rA + sA$
f. $r(sA) = (rs)A$

Properties of matrix multiplication

Properties of Matrix Multiplication

The following theorem lists the standard properties of matrix multiplication. Recall that I_m represents the $m \times m$ identity matrix and $I_m\mathbf{x} = \mathbf{x}$ for all $\mathbf{x}$ in $\mathbb{R}^m$.

Let A be an $m \times n$ matrix, and let B and C have sizes for which the indicated sums and products are defined.

a. $A(BC) = (AB)C$ (associative law of multiplication)
b. $A(B + C) = AB + AC$ (left distributive law)
c. $(B + C)A = BA + CA$ (right distributive law)
d. $r(AB) = (rA)B = A(rB)$ for any scalar r
e. $I_mA = A = AI_n$ (identity for matrix multiplication)

Warnings about matrix multiplication

WARNINGS:

1. In general, $AB \neq BA$.

2. The cancellation laws do *not* hold for matrix multiplication. That is, if $AB = AC$, then it is *not* true in general that $B = C$. (See Exercise 10.)

3. If a product AB is the zero matrix, you *cannot* conclude in general that either $A = 0$ or $B = 0$. (See Exercise 12.)

Multiplicative powers of matrices

Note: $A^0 = I_n$

Powers of a Matrix

If A is an $n \times n$ matrix and if k is a positive integer, then A^k denotes the product of k copies of A:

$$A^k = \underbrace{A \cdots A}_{k}$$

If A is nonzero and if $\mathbf{x}$ is in $\mathbb{R}^n$, then $A^k\mathbf{x}$ is the result of left-multiplying $\mathbf{x}$ by A repeatedly k times. If $k = 0$, then $A^0\mathbf{x}$ should be $\mathbf{x}$ itself. Thus A^0 is interpreted as the identity matrix. Matrix powers are useful in both theory and applications (Sections 2.6, 4.9, and later in the text).

Algebra and the transpose: basic facts.

Let A and B denote matrices whose sizes are appropriate for the following sums and products.

a. $(A^T)^T = A$
b. $(A + B)^T = A^T + B^T$
c. For any scalar r, $(rA)^T = rA^T$
d. $(AB)^T = B^TA^T$